Problem 1.46

(a) Show that

$$x\frac{d}{dx}(\delta(x)) = -\delta(x).$$

[*Hint:* Use integration by parts.]

(b) Let $\theta(x)$ be the (**Heaviside**) **step function**:

$$\theta(x) \equiv \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{1.95}$$

Show that $d\theta/dx = \delta(x)$.

Solution

Part (a)

Consider the integral of $x\delta'(x)$ over the whole line and integrate by parts.

$$\begin{aligned} \int_{-\infty}^{\infty} x\frac{d}{dx}[\delta(x)]\,dx &= x\delta(x)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx}(x)\delta(x)\,dx \\ &= (0)\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(1)\delta(x)\,dx \\ &= \int_{-\infty}^{\infty}[-\delta(x)]\,dx \end{aligned}$$

Therefore, matching the integrands,

$$x\frac{d}{dx}[\delta(x)] = -\delta(x).$$

Part (b)

By definition, the delta function $\delta(x)$ satisfies the following two properties.

$$(1) \quad \delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

$$(2) \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Show that $d\theta/dx$ satisfies property (1). Note that because $0 = \lim_{x\to 0^-} \theta(x) \neq \lim_{x\to 0^+} \theta(x) = 1$, the derivative of the step function is not defined at $x = 0$. But since $\theta(x)$ jumps from 0 to 1 at $x = 0$, the slope can be thought of as $+\infty$ there.

$$\frac{d\theta}{dx} = \begin{cases} \frac{d}{dx}(0) & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ \frac{d}{dx}(1) & \text{if } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Show that $d\theta/dx$ satisfies property (2).

$$\int_{-\infty}^{\infty} \frac{d\theta}{dx}\, dx = \theta(\infty) - \theta(-\infty)$$

$$= 1 - 0$$

$$= 1$$

Therefore,

$$\frac{d\theta}{dx} = \delta(x).$$